Quantum Groups Seminar 17 March, 2025

Quantum Graphs, Subfactors, & Tensor Categories

j/w M. Brannan, arXiv: 2409.01951

robertohp.math@gmail.com

Intro: Goal: Framework for equivariant quantum graphs.

Motivation:

Quantum Information:

- Zero-error Communication [DuSeWi 16]
- nonlocal games [MRV 18 & 19, Brannan et al, Gold 24]

\+

Operator Algebra:

- Cayley graph quantum group [Ver 05, Was 23]
- Quantum Relations [Weaver 12, 21]

Point: All quantum graphs are equivariant wrt some quantum symmetry.

Thm A [B-HP24] Every "quantum graph" is modeled over $A \overset{1}{\underset{E}{\subseteq}} B$.

c.f. [Frucht '39] $\forall\ \Gamma$: fin gp $\exists$ simple undirected finite graph G, $\Gamma \simeq \mathrm{Aut}(G)$.

Thm B [B-HP24]: Every quasitriangular finite quantum groupoid arises as "crossing quantum automorphisms" of some quantum graph.

Q: ¿How to categorify a graph?

Classically: $\mathcal{G} = \Big(\underbrace{\overbrace{\mathbb{C}^V \xrightarrow{\omega} \mathbb{C}}}_{B}, \ \underbrace{\hat{T} \in \mathrm{Mat}_V(\{0,1\})}_{\text{adjacency operator}} \Big)$

simple, finite weighted classical graph

Problems:
- Quantize finite sets
- Characterize adjacency endomorphisms of B.

Higher structure on sets: $B \curvearrowright L^2(B,\omega) \ (= B\Omega)$

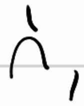
 : $L^2B \longrightarrow \mathbb{C}$, $b\Omega \longmapsto \omega(b)$

 : $L^2B \otimes L^2B \longrightarrow L^2B$, $b_1\Omega \otimes b_2\Omega \longmapsto b_1 \cdot b_2 \Omega$

adjoints

 : $\mathbb{C} \longrightarrow L^2B$

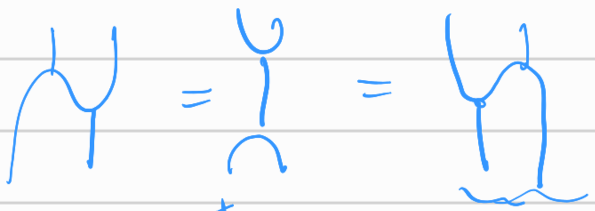
 : $L^2B \longrightarrow L^2B \otimes L^2B$ $(b \longmapsto b \otimes b)$

Q-System ⇝ $Q = \Big(\underbrace{L^2B,}$ , , ,  $\Big)$

(co)assoc + (co) unital $*$-algebra

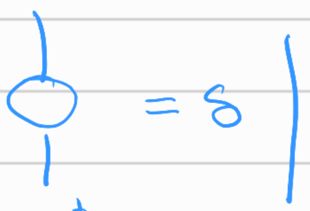
$m^\dagger$ is B–B

Frobenius Algebra +



$m^\dagger \circ m$ $\qquad (\mathrm{id}_B \otimes m) \circ (m^\dagger \otimes \mathrm{id}_B)$

"δ-form" +



$= \delta \ | = \delta \, \mathrm{id}_B$

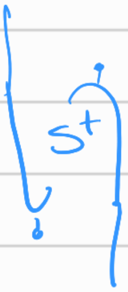
$m \circ m^\dagger$

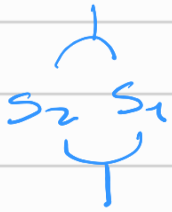
Dfn: $\left\{ \begin{array}{c} \text{finite} \\ \text{quantum} \\ \text{set} \end{array} \right\} \longleftrightarrow \{\text{Q-System}\}$

Key Obs: $End(Q)$ has 2 C^*-alg structures:

$s_1, s_2 \in End(Q)$ • $(\circ, \dagger)$: composition $s_2 \circ s_1 =$ 

$\in End(Q)$

$s^* =$ 

•• $(\star, *)$: convolution $s_2 \star s_1 =$ 

$\in End(Q)$

$m \circ (s_2 \otimes s_1) \circ m^\dagger$

Eg: $Q = (\mathbb{C}^n, tr)$ $(s_1 \star s_2)^{ij} = s_1^{ij} \cdot s_2^{ij}$

$\therefore$ $\hat{T} \in End(Q)$ is adj. O.P.
$\iff$ $\hat{T} = \hat{T} \star \hat{T}$

Dfn: f.d. quantum graph:

$$\mathcal{G} = \Big(\underbrace{\mathbb{C} \overset{\omega}{\subset} B}_{Q\text{-sys}},\ \hat{T} \star \hat{T} = \hat{T} \in End(B)\Big).$$

$Rep_f(B)$

Categorified graphs: $(\mathcal{C}, \circ, \oplus, \simeq, \dagger, \otimes, 1_\mathcal{C}, \bar{\cdot})$

Unitary Tensor Category

Dfn: $\mathcal{C}$-equivariant Quantum Graph

$$\mathcal{G} = \big(Q \in \mathcal{C},\ \hat{T} \star \hat{T} = \hat{T} \in End(Q)\big)$$

Complete Graph

Eg: $\mathbb{K}_Q = (Q,$  $)$ "all ones matrix"

 $\{ Triv_Q = (Q, \underset{id_Q}{|})$ "trivial Q-graph"

Eg: $\mathcal{C} = \mathrm{Hilb}_f$ "linear algebra"

$$\{Q\text{-Systems}\} \longleftrightarrow \{\text{f.d. } C^*\text{-alg} + \text{state}\}$$

$$\mathcal{G} = (\cancel{\text{abelian}}\ Q + \hat{T}_{ij} \in \{0,1\}) : \text{classical graph}$$

$\approx \mathbb{C}^V$

$$\mathcal{G} = (Q + \hat{T} * \hat{T} = \hat{T}) : \text{f.d. Qu Graph}$$

$Q = \bigoplus M_{n_i}$

RMK: cl-Hilb_f graphs $\overset{[\text{Kauff Kup}]}{\longleftrightarrow}$ Knot invariants

OMR ¿Qu Invariants from Qu Graphs?

Eg: $\mathcal{C} = \mathrm{Rep}_f(G)$ "$\mathcal{C}$ - equivariant"

$$\{Q\} \longleftrightarrow \{\text{f.d. } G\text{-}C^*\text{-alg} + G\text{-state}\}$$

Pontryagin Duality

$$\mathcal{G} = (Q, \hat{T}) \underset{[\text{Verdon 23}]}{\longleftrightarrow} \text{"G-Covariant qu Channel"}$$

$$\mathcal{C} = \mathrm{Hilb}_f(\Gamma)$$

$$\{Q\} \longleftrightarrow \{\Lambda \leq \Gamma + \cancel{2\text{-cocycle}}\}$$

$$\mathcal{G} = (\ell^\infty \Lambda, S \subseteq \Lambda) \longleftrightarrow \mathrm{Cay}(\Lambda, S)$$

$\mathcal{C}$-equivariant graphs

Eg: $R = N \subset M = R$ $\quad [M:N] \in \{4\cos^2 \pi/n\}_{n \geq 3}$

e

An hyperfinite subfactor

$$\mathbb{K}_M = (M, \underset{\text{Jones projection}}{e})$$

Tracial regular graph non integral degree $4\cos^2 \pi/n$

Q: ¿Where do $\hat{T} \star \hat{T} = \hat{T}$ come from?

Key insights from subfactors!

Facts • Every $\mathcal{C}$ is "Concrete" [HHP20]

$$\mathcal{C} \subseteq \mathrm{Bim}_f(\underset{\sim}{A}) \Big\} \; M_n(A)$$

↖ unital simple monotracial

"linear algebra with coefficients in A"

•• "C^*_1-algebras are Q-System complete" [CHPJP22]

$Q \simeq {}_A B_A$

$$\{\underset{\sim}{Q \in \mathcal{C}}\} \rightleftarrows \{\, \underset{\sim}{A \overset{E}{\subset} B} : \underset{\text{finite index}}{\underbrace{B_A \cong p[A^{\oplus n}]}} \,\}$$

$$\therefore\ G := \Big(\underset{\text{S-form}}{{}_A B_A}, \; \hat{T} \star \hat{T} = \hat{T} \in \underset{\text{f.d. } C^*\text{-algs!}}{\mathrm{End}({}_A B_A)} \Big)$$

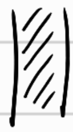
always concrete

Connes's fusion rel ⊗-product

&

Q is shaded "pair of pants"

$$Q = {}_A B_A \simeq {}_A \bar{B} \boxtimes_B B_A$$

$\mathrm{id}_Q =$ Q $\simeq$  $= \mathrm{id}_{\bar{B} \boxtimes B}$

$\bar{B} \boxtimes B$

 $\longmapsto$ $m \simeq$ 

$E \simeq$  $\longleftarrow$ 

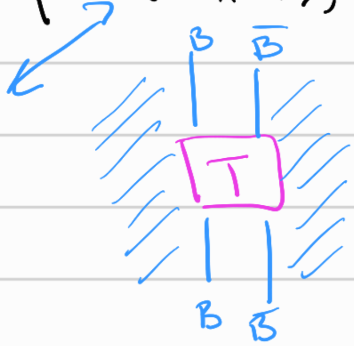
Obs: UTCs come in pairs

$$\underbrace{\mathcal{C}_{ACB}}_{A\subset B\subset B_1} = \langle {}_A \bar{B} \boxtimes_B B_A \rangle \longleftrightarrow \langle {}_B B \boxtimes_A \bar{B}_B \rangle = \mathcal{C}_{BCB_1}$$

${}_A B_A$ $\qquad$ $\underbrace{}_{B\subset B_1}$ "Jones basic construction"

Quantum Fourier Transform:

$$\mathcal{F}: \left(\mathrm{End}({}_B B \boxtimes_A \bar{B}_B), \circ\right) \xleftrightarrow{C^*\text{-isom.}} \left(\mathrm{End}({}_* \bar{B} \boxtimes_B B), *\right)$$



$\mapsto \hat{T} :=$

$$\left\{ \boxed{T \circ T = T} \right\} \longleftrightarrow \left\{ \hat{T} * \hat{T} = \hat{T} \right\}$$

Cor: All quantum graphs arise from "dual inclusion" via $\mathcal{F}$.

Thm/Dfn: $\mathcal{G} := \left({}_A B_A, \underbrace{T\circ T = T}_{\text{edge projector}} \in \mathrm{End}({}_B B_B)\right)$

complete graph

$\mathcal{C}$-Tholt's Thm: Every quasitriangular finite quantum groupoid arises as "crossing quantum automorphisms" of some quantum graph.

$$\underline{\mathbb{H}} = (H, m, 1, \Delta, \epsilon, S) \longleftrightarrow \underline{\mathcal{C}} = \mathrm{Rep}_f(\mathbb{H})$$

"fusion category" i.e. finite

quasitriangular $\longleftrightarrow$ $\exists\, \underline{\beta} =$

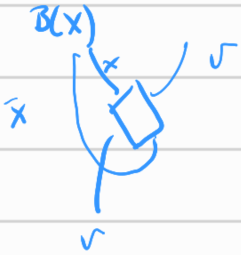
Reconstruction: [Ni-Vai 02]

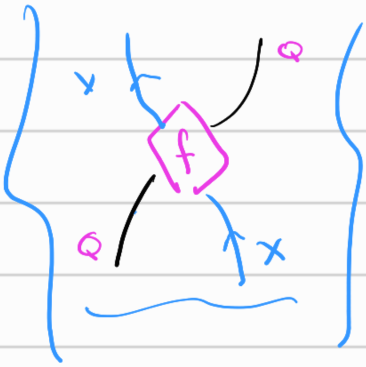
$\{\mathbb{H}\} \rightleftarrows \{ N \subset M : \text{finite-depth, hyperfinite, finite-index} \} \to \mathcal{C}_{N\subset M}$ finite, $N \simeq R \simeq M$

Graph Quantum Automorphisms:

Hilb

$B(X)$



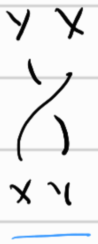


$\cong_{\otimes}$ [MRV 18]

$\mathrm{Rep}(\mathrm{Aut}^+(\text{classical graph}))$

"Random function over X-PVM"

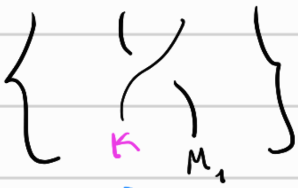
$Q = \mathbb{C}^V, \quad \hat{T}_{ij} \in \{0,1\}$

Key: $\beta =$  is Qu ~~Function~~ Automorphisms in $\mathrm{Rep}(\mathbb{H}) = \mathcal{C}$!

Pf sketch Thm B:

• $\mathbb{H} \rightsquigarrow \mathcal{C}_{M\subset M_1} \cong_{\otimes} \mathrm{Rep}(\mathbb{H})$

• • $\mathcal{G} := \mathbb{K}_{M_1} = ({}_M {M_1}_M, e)$

• • • $\mathcal{C}_{M\subset M_1} \underset{\otimes}{\overset{\text{For}}{\cong}} \{$  $\}_{K\in\mathcal{C}}$ Quantum Auto of $\mathbb{K}_{M_1}$

${}_M K_M$

Frucht's

∴ $\mathrm{Rep}(\mathbb{H})$ is Crossing Qu-Isomorphisms of $\mathcal{G}$.

$\mathcal{C} = D(\mathcal{C}') \hookrightarrow \mathcal{B}_{\mathcal{C},M}$

Thank You!